d-outcome measurement for a nonlocality test

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Abstract
For the purpose of a nonlocality test, we propose a general correlation
observable of two parties by utilizing local d-outcome measurements with
$SU(d)$ transformations and classical communications. Generic symmetries
of the $SU(d)$ transformations and correlation observables are found for the
test of nonlocality. It is shown that these symmetries dramatically reduce
the number of numerical variables, which is important for numerical analysis
of nonlocality. A linear combination of the correlation observables, which is
reduced to the Clauser–Horne–Shimony-Holt (CHSH) Bell's inequality for two
outcome measurements, leads to the Collins–Gisin–Linden–Massar–Popescu
(CGLMP) nonlocality test for d-outcome measurement. As a system to be
tested for its nonlocality, we investigate a continuous-variable (CV) entangled
state with d measurement outcomes. It allows the comparison of nonlocality
based on different numbers of measurement outcomes on one physical system.
In our example of the CV state, we find that a pure entangled state of any
degree violates Bell's inequality for $d (\geqslant 2)$ measurement outcomes when the
observables are of $SU(d)$ transformations.

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1. Introduction

Nonlocality is one of the most profound aspects of a quantum mechanical system and it is a
fundamental resource for quantum information processing. Nonlocality has been studied
commonly in the operational perspective based on Bell's inequalities for bipartite two-
dimensional systems with dichotomic measurements. The extensions to arbitrary dimensional
systems have been proposed [1–3]. Recently, Kaszlikowski *et al* [2] considered joint
probabilities of two distant measurements and suggested how to compare the strength of

nonlocality between different dimensional systems for different numbers of measurement outcomes. A maximally entangled pure system violates Bell's inequality but as enough white noise is added the system loses its nonlocality. Kaszlikowski *et al* proposed the fraction of white noise as a measure of nonlocality, which may be used for its cross-dimensional comparison. Their numerical analysis for a maximally entangled state showed that the degree of violation increases monotonically with respect to the number of outcomes.

More recently, Collins *et al* [3] developed a family of Bell's inequalities for an arbitrary finite number of measurement outcomes. The family of inequalities are in good agreement with Kaszlikowski *et al*'s results in terms of their measure of nonlocality. These studies would imply that the critical fraction of white noise is a useful measure in comparing the amounts of nonlocality for the different dimensional systems. The measure of nonlocality based on the noise fraction is, on the other hand, criticized by Acin *et al* [4] as they found that partially entangled systems can give a larger violation (or stronger resistance to noise) of Bell's inequality than the maximally entangled state. The other approach for a substantial violation of local realism was introduced by van Dam *et al* [5] and they found that CHSH inequality is the strongest nonlocality test for a bipartite system in terms of the statistical strength.

In the test of Bell's inequality, a set of unitary transformations play an important role in the violation of Bell's inequality because local measurement settings for each party are characterized by local unitary transformations. In earlier studies [2–4], the transformations are restricted to quantum Fourier transformations (QFT). For a d-dimensional system the most general unitary transformation forms the group $SU(d)$. It is thus questionable whether the QFT is sufficient to fully reveal quantum nonlocality. It has been known that the QFT is sufficient for a maximally entangled system of $d = 2$ and 3 [2]. However, it is still an open question for other dimensional systems and, more importantly, for an arbitrarily entangled system. This question is investigated in this paper.

Quantum nonlocality for continuous-variable (CV) systems has been studied in various contexts. Bell argued that the original Einstein–Podolsky–Rosen (EPR) state [7] would not violate Bell-like inequalities since it has a positive-definite Wigner function and thus its correlation function with respect to position and momentum observables can be simulated by local hidden variables. On the other hand, introducing dichotomic measurements such as even or odd parities of the photon number and presence or absence of photons, Banaszek and Wódkiewicz [8, 9] showed the nonlocality of the EPR state. The measurements follow displacement operations, that is, translations in the phase spaces of the modes. However, the scheme by Banaszek and Wódkiewicz did not give a maximal violation for the inequality. This motivated Chen *et al* [10] to investigate another type of observable, with the unitary transformations other than the displacement in phase space, which results in the maximal violation for EPR state. The observable, so-called 'pseudo-spin' operator, is defined as tensor summations of Pauli spin operators, which is an element of an $SU(2)$ group.

The generalization of a dichotomic measurement to an arbitrary finite number of outcomes for a nonlocality test of a CV state was proposed by Brukner *et al* [12] as establishing a correspondence between a CV and a discrete system of an arbitrary finite dimension. However, in their work, it is not clear whether the correspondence can be given as a physically plausible map, i.e. a completely positive (CP) map. Moreover, in their analysis for nonlocality, Brukner *et al* did not employ general transformations in $SU(d)$ but the simple QFT transformations in varying the configuration of measurements. Thus, the question of QFT being sufficient to reveal nonlocality arises in CV systems as well as in finite-dimensional systems.

In this paper, we formulate Bell's inequality in terms of a linear summation of correlation functions which utilize the most general projective d-outcome measurements. For the correlation function, we introduce a general form of correlation observable between two

d-level systems and find the eigenvalues from the generic conditions that any correlation function should satisfy. For the observable, we exploit all the possible unitary transformations in the $SU(d)$ group on the configurations of local d-outcome measurements. The subgroup algebra of $SU(d)$ allows us to prove that $(d^2 - d)$ number of real parameters are sufficient to describe the local unitary operation. By inspecting symmetries and performing numerical analysis, we show, while the QFT suffices for a maximally entangled system, a partially entangled system requires more general transformations in order to fully investigate its nonlocality.

Bell's inequalities are applied to a CV state whose infinite-dimensional Hilbert space is decomposed onto the tensor sum of d-dimensional subspaces. The decomposition maps any CV state onto an arbitrary finite-dimensional state. We prove that the mapping is linear, trace preserving, complete positive. After applying the mapping, we investigate the violation of Bell's inequality for the two-mode squeezed vacuum state, as an example of a CV state, with the different outcome measurements. To search for the optimal violation in $SU(d)$ transformation, several numerical methods are assessed.

2. Bell's inequalities with d-outcome measurements

In this section, we investigate Bell's inequalities by considering the $SU(d)$ group of transformations for the measurement with d outcomes. The series of inequalities may be derived by introducing 'classically correlated observables' which can be constructed by local measurements and classical communications.

2.1. Special unitary transformation for the d-outcome measurement

A measurement of a system is represented by a Hermitian operator which is called an observable. Any Hermitian operator on a d-dimensional Hilbert space $\mathcal{H}_d$ can be expanded by the identity operator and the group generators of $SU(d)$ algebra. Such a typical description in terms of group generators was introduced by Hioe and Eberly [14]. In order to obtain the generators of the $SU(d)$ group, one may introduce transition-projection operators

$$\hat{P}_{jk} = |j\rangle\langle k|, \tag{1}$$

where $\{|j\rangle\}$ is an orthonormal basis set on $\mathcal{H}_d$. Now, the $(d^2 - 1)$ Hermitian operators are constructed as

$$\hat{u}_{jk} = \hat{P}_{jk} + \hat{P}_{kj} \tag{2}$$

$$\hat{v}_{jk} = i(\hat{P}_{jk} - \hat{P}_{kj}) \tag{3}$$

$$\hat{w}_l = -\sqrt{\frac{2}{l(l+1)}} \left(\sum_{i=1}^{l} \hat{P}_{ii} - l\hat{P}_{l+1l+1} \right), \tag{4}$$

where $1 \leqslant l \leqslant d-1$ and $1 \leqslant j < k \leqslant d$. It is easy to check that when $d = 2$ these generators are Pauli spin operators.

The set of $G = \{\hat{u}_{12}, \hat{u}_{13}, \ldots, \hat{v}_{12}, \hat{v}_{13}, \ldots, \hat{w}_1, \ldots, \hat{w}_{d-1}\}$ is composed of generators for $SU(d)$ group, fulfilling the relations of tracelessness $\text{Tr}(\hat{s}_j) = 0$ and orthogonality $\text{Tr}(\hat{s}_i\hat{s}_j) = 2\delta_{ij}$ for $\hat{s}_i, \hat{s}_j \in G$. The elements $\hat{s}_i \in G$ hold the algebraic relation,

$$[\hat{s}_j, \hat{s}_k] = 2i \sum_l f_{jkl}\hat{s}_l, \tag{5}$$

where f_{jkl} is the antisymmetric structure constant of the $SU(d)$ algebra.

The set G can be divided into the three mutually exclusive subsets of operators: $U = \{\hat{u}_{jk}\}$, $V = \{\hat{v}_{jk}\}$ and $W = \{\hat{w}_l\}$, which contain $d(d-1)/2$, $d(d-1)/2$ and $(d-1)$ elements respectively. The operators in each subset satisfy the algebraic relations

$$[\hat{u}_{ij}, \hat{u}_{kl}] = -\mathrm{i}\{\delta_{jl}(1-\delta_{ik})\hat{v}_{ik} + \delta_{ik}(1-\delta_{jl})\hat{v}_{jl} + \delta_{jk}(1-\delta_{il})\hat{v}_{il} - \delta_{il}(1-\delta_{kj})\hat{v}_{kj}\},$$

$$[\hat{v}_{ij}, \hat{v}_{kl}] = -\mathrm{i}\{\delta_{jl}(1-\delta_{ik})\hat{v}_{ik} + \delta_{ik}(1-\delta_{jl})\hat{v}_{jl} - \delta_{jk}(1-\delta_{il})\hat{v}_{il} + \delta_{il}(1-\delta_{kj})\hat{v}_{kj}\}, \tag{6}$$

$$[\hat{w}_i, \hat{w}_j] = 0.$$

The commutators between the elements from U or V result in the operators in the subset V while any elements in W commute to each other. The commutation relations among the operators from the different subsets can be found as

$$[\hat{u}_{ij}, \hat{w}_l] = -(x_{il} - x_{jl})\hat{v}_{ij}$$

$$[\hat{v}_{ij}, \hat{w}_l] = -\mathrm{i}(x_{il} - x_{jl})\hat{u}_{ij} \tag{7}$$

$$[\hat{u}_{ij}, \hat{v}_{kl}] = \mathrm{i}(\delta_{jk}\hat{u}_{il} + \delta_{ik}\hat{u}_{jl} - \delta_{jl}\hat{u}_{ik} - \delta_{il}\hat{u}_{kj}) + 2\mathrm{i}\delta_{ik}\delta_{jl}(|j\rangle\langle j| - |i\rangle\langle i|),$$

where the coefficient x_{il} is given by

$$x_{il} = -\sqrt{\frac{2}{l(l+1)}}\left(\sum_{k=1}^{l}\delta_{ik} - l\delta_{il+1}\right) \tag{8}$$

and δ is the Kronecker delta function. As a summary, one may symbolically express the commutation relation among the subsets U, V and W as

$$\begin{array}{lll} [U, U] \propto V, & [V, V] \propto V, & [W, W] \propto 0 \\ [U, W] \propto V, & [V, W] \propto U, & [U, V] \propto U + W. \end{array} \tag{9}$$

Using the $SU(d)$ group generators, any Hermitian operator on the d-dimensional Hilbert space is represented by

$$\hat{\Omega}(\vec{a}) = \frac{a_0}{d}\hat{\mathbb{1}} + \frac{1}{2}\sum_{j=1}^{d^2-1}a_j\hat{s}_j, \tag{10}$$

where $a_0 = \mathrm{Tr}\,\hat{\Omega}(\vec{a})$ and $a_j = \mathrm{Tr}\,\hat{s}_j\hat{\Omega}(\vec{a})$ are real numbers due to the hermiticity condition of the observable $\hat{\Omega}(\vec{a})$. The coefficient a_j comprises a (d^2-1)-dimensional vector $\vec{a} = (a_1, \ldots, a_{d^2-1})$ which we call a *generalized Bloch vector*, while a_0 is constant over any $SU(d)$ transformations.

In the Heisenberg picture, the unitary transformation $\hat{U}$ of the Hermitian operator,

$$\hat{\Omega}(\vec{a}) \rightarrow \hat{\Omega}(\vec{a}') = \hat{U}\hat{\Omega}(\vec{a})\hat{U}^\dagger, \tag{11}$$

can also be described as a transformation of the generalized Bloch vector $\vec{a}$. Decomposing $\hat{\Omega}(\vec{a}')$ in the form (10) with coefficients a_j' and using the invariance of the trace under cyclic permutation, the components a_j' of the transformed generalized Bloch vector are found to be

$$a_j' = \mathrm{Tr}(\hat{s}_j\hat{\Omega}(\vec{a}')) = \mathrm{Tr}(\hat{U}^\dagger\hat{s}_j\hat{U}\hat{\Omega}(\vec{a})). \tag{12}$$

Since $\hat{U}^\dagger\hat{s}_j\hat{U}$ is also Hermitian and traceless, it can be expanded in terms of the $SU(d)$ generators as

$$\hat{U}^\dagger\hat{s}_j\hat{U} \equiv \sum_k T_{jk}\hat{s}_k, \tag{13}$$

where $T_{jk} = \frac{1}{2}\mathrm{Tr}(\hat{U}^\dagger\hat{s}_j\hat{U}\hat{s}_k)$ is an element of a $(d^2-1)\times(d^2-1)$ real matrix. The matrix T represents the direct relation between the transformed and untransformed generalized Bloch

vectors $a'_k = \sum_j T_{jk} a_j$. As the norm of the generalized Bloch vector remains constant under the transformation, the real matrix T is orthogonal.

An operator $\hat{U}$ in $SU(d)$ can be represented in terms of the group generators $\vec{s} = (\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_{d^2-1})$ as

$$\hat{U}(\vec{p}) = \exp(-i\vec{p} \cdot \vec{s}), \tag{14}$$

where $\vec{p}$ is a $(d^2 - 1)$-dimensional parameter vector. The parametrization in equation (14) is said to be canonical. Experimentally, for the optical device, it is possible to realize the discrete unitary operation in $SU(d)$ using biased multiport beam splitters [15]. In order to derive the explicit matrix elements of T in equation (13) corresponding to the unitary operator $\hat{U}(\vec{p})$, one may consider a set of differential equations for the generators:

$$\frac{\partial}{\partial t}\hat{s}_j(t) = \hat{U}^\dagger(t\vec{p}) \left\{ i \sum_k p_k[\hat{s}_k, \hat{s}_j] \right\} \hat{U}(t\vec{p})$$

$$= \sum_l \left(-2 \sum_k p_k f_{kjl} \right) \hat{s}_l(t), \tag{15}$$

where $\hat{s}_j(t) = \hat{U}^\dagger(t\vec{p})\hat{s}_j\hat{U}(t\vec{p})$. After solving the differential equation and setting $t = 1$, the matrix T is derived in terms of the parameter vector $\vec{p}$ and the antisymmetric structure constant f_{kjl} as

$$T(\vec{p}) = \exp(-2F(\vec{p})), \qquad \text{where} \quad F_{jl}(\vec{p}) = \sum_k p_k f_{kjl}. \tag{16}$$

The antisymmetric characteristics of the structure constant f_{kjl} are related with the orthogonality of T as $T^T T = TT^T = \mathbb{1}_{d^2-1}$ where $\mathbb{1}_{d^2-1}$ is an identity matrix on $(d^2 - 1)$-dimensional vector space.

It is notable that a commutation relation appears in equation (15). From the fact that the group generators $\{\hat{s}_j\}$ are divided into three subsets U, V and W and the generators of each subset satisfy the algebraic relations (6) and (7), one can find some symmetries for the rotation of generators. In particular, since $[\hat{w}_i, \hat{w}_j] = 0$, it is possible to find that the rotation of generators $\{\hat{w}_l\} \in W$ along the direction of any $\hat{w}_j$ results in the generator itself as

$$\exp(-ip_{w_j}\hat{w}_j)\hat{w}_l \exp(ip_{w_j}\hat{w}_j) = \hat{w}_l, \tag{17}$$

where p_{w_j} is the $\hat{w}_j$ component of the parameter vector appeared in equation (14).

Equation (17) implies that the dimensionality of the nontrivial parameter vector $\vec{p}$ for the unitary transformation of the Hermitian operator (11) can be reduced to $(d^2 - d)$. Without loss of generality, the Hermitian operator (10) can be written with a given orthogonal basis $\{|j\rangle\}$ and the unitary operator of the basis transformation as

$$\hat{\Omega}(\vec{a}) = \hat{U}(\vec{p}_a) \sum_{j=1}^d \Omega_j |j\rangle\langle j| \hat{U}^\dagger(\vec{p}_a), \tag{18}$$

where Ω_j is the nondegenerate eigenvalue. The generators in the subset W are sufficient to reconstruct all the diagonal bases $\{|j\rangle\langle j|\}$, that is,

$$|j\rangle\langle j| = \frac{1}{d}\hat{\mathbb{1}} - \sum_{k=0}^{d-j} g_k^j \hat{w}_{j-1+k}, \tag{19}$$

where $\hat{g}_k^j = (1 - j\delta_{k0})\sqrt{\frac{1}{2(j+k)(j+k-1)}}$. With the help of equations (17) and (19), one can see that the dimensionality of the nontrivial parameter vector $\vec{p}_a$ in equation (18) is $(d^2 - d)$. This implies that any unitary transformation in $SU(d)$ for the observable in the d-dimensional Hilbert space is sufficient with $(d^2 - d)$ number of real parameters.

2.2. Classically correlated observable with d-outcome measurement

Two observers, say Alice and Bob, perform local measurements on their own d-level systems and they communicate their outcomes via a classical channel. A classically correlated observable is thus constructed as assigning a weight $\mu_{i,j}$ for the pair of outcomes, i and j:

$$\hat{E} = \sum_{i,j=1}^{d} \mu_{i,j} |i\rangle_a \langle i| \otimes |j\rangle_b \langle j|. \tag{20}$$

The correlation coefficient matrix μ is a $d \times d$ real matrix. It is notable that equation (20) is the most general form of a correlation measure between any two d-level systems. The correlation observable $\hat{E}$ involves d^2 correlation coefficients. We show that without loss of generality, the number of independent parameters reduces to d and we determine their values.

We require that a correlation observable should satisfy the following conditions:

C.1. A correlation function should be indifferent to local polarization, which means that

$$\mathrm{Tr}\,\hat{E}\hat{\rho}_A \otimes \mathbb{1}_B = \mathrm{Tr}\,\hat{E}\mathbb{1}_A \otimes \hat{\rho}_B = 0, \tag{21}$$

where $\hat{\rho}_{A,B} = \mathrm{Tr}_{B,A}\,\hat{\rho}$ are the reduced density operators. This raises the following condition:

$$\sum_j \mu_{i,j} = 0, \forall\, i \qquad \text{and} \qquad \sum_i \mu_{i,j} = 0, \forall\, j. \tag{22}$$

For the case of two outcomes, there is the well accepted correlation matrix $\mu = \{\{\mu_{1,1}, \mu_{1,2}\}, \{\mu_{2,1}, \mu_{2,2}\}\} = \{\{1, -1\}, \{-1, 1\}\}$. Here, the translational symmetry, $\mu_{1,1} = \mu_{2,2}$ and $\mu_{12} = \mu_{21}$, and equal spacing, $\mu_{1,1} - \mu_{2,1} = 2$ leads the correlation observable to optimize the measure of correlation. We generalize these in the following two conditions.

C.2. The correlation coefficients are unbiased over their outcomes (translational symmetry within modulo d):

$$\mu_{i+k,j+k} = \mu_{i,j}, \qquad \forall\, k. \tag{23}$$

C.3. The coefficients are equally separated and normalized (maximal discrimination):

$$\mu_{i,j} - \mu_{i+1,j} = \frac{2}{d-1}, \qquad \text{for} \quad i \geqslant j. \tag{24}$$

The condition C.2 leads the correlation matrix μ to be in the form of

$$\mu = \begin{pmatrix} \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_d \\ \mu_d & \mu_1 & \mu_2 & \cdots & \mu_{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_2 & \mu_3 & \mu_4 & \cdots & \mu_1 \end{pmatrix} \tag{25}$$

and further the condition C.1 implies that

$$\sum_l \mu_l = 0. \tag{26}$$

The condition C.3 determines all the μ_l such that $\mu_1 = 1$ for a maximally correlated state, $\mu_d = -1$ for a maximally anti-correlated state and the other μ_l are assigned to have equally

spaced values between 1 and -1. Thus, the three conditions C.1, C.2 and C.3 uniquely determine the correlation matrix μ,

$$\mu_{i,j} = 1 - 2\frac{(i-j) \bmod d}{d-1}. \tag{27}$$

Using equation (19), the correlation observable $\hat{E}$ in equation (20) can be written in terms of the $SU(d)$ generators as

$$\hat{E} = \sum_{k,l=1}^{d-1} \tilde{\mu}_{k,l} \hat{w}_k \otimes \hat{w}_l, \tag{28}$$

where $\tilde{\mu}_{k,l}$ is the transformed correlation matrix from $\mu_{j,k}$. That is,

$$\tilde{\mu}_{k,l} = \sum_{i=1}^{k+1} \sum_{j=1}^{l+1} g_{k-i+1}^i g_{l-j+1}^j \mu_{i,j}, \tag{29}$$

where g_k^i is given in equation (19). Note that the correlation observable $\hat{E}$ in equation (28) does not contain any local identity operator $\mathbb{1}_d$ due to the condition C.1. Further, the observable transformed by local unitary operations is written as

$$\begin{aligned}
\hat{E}(\vec{p}, \vec{q}) &= \hat{U}(\vec{p}) \otimes \hat{U}(\vec{q}) \hat{E} \hat{U}^\dagger(\vec{p}) \otimes \hat{U}^\dagger(\vec{q}) \\
&= \sum_{l,m=1}^{d^2-1} \tilde{\mu}_{l,m}(\vec{p}, \vec{q}) \hat{s}_l \otimes \hat{s}_m,
\end{aligned} \tag{30}$$

where $\tilde{\mu}(\vec{p}, \vec{q}) = T^T(\vec{p})\tilde{\mu}T(\vec{q})$. The unitary operators $\hat{U}(\vec{p})$ and $\hat{U}(\vec{q})$ determine the measurement configuration for each side. Without any constraint for the d-outcome measurement, the unitary operators are subjected to the $SU(d)$ group.

2.3. Bell's inequalities for a bipartite d-dimensional system

In order to investigate nonlocality of a bipartite system, we introduce a Bell function which can be constructed by a linear combination of correlation functions of two parties. The Bell function can be written without loss of generality as

$$B = \sum_i c_i E(\vec{p}_i, \vec{q}_i), \tag{31}$$

where the correlation function $E(\vec{p}, \vec{q}) = \text{Tr}\,\hat{E}(\vec{p}, \vec{q})\hat{\rho}$ and $\vec{c} = \{c_i\}$ is an arbitrary vector which satisfies a normalized condition $\sum_i c_i = 2$ to make the Bell function B a polytope [16]. Note that the correlation function $E(\vec{p}, \vec{q}) \in [-1, 1]$ for all $\vec{p}$ and $\vec{q}$. The classically correlated observable $\hat{E}(\vec{p}, \vec{q})$ can be written as

$$\hat{E}(\vec{p}, \vec{q}) = \sum_{i,j} \mu_{i,j} \hat{P}_i(\vec{p}) \otimes \hat{P}_j(\vec{q}), \tag{32}$$

where the projector $\hat{P}_i(\vec{p}) = \hat{U}(\vec{p})|i\rangle\langle i|\hat{U}^\dagger(\vec{p})$ is for the ith outcome with the measurement configuration $\vec{p}$ and the correlation matrix μ is given in equation (27). The joint probability that Alice and Bob obtain the outcomes i and j with the measurement configurations $\vec{p}$ and $\vec{q}$ is given by

$$P_{ij}(\vec{p}, \vec{q}) = \text{Tr}(\hat{P}_i(\vec{p}) \otimes \hat{P}_j(\vec{q})\hat{\rho}). \tag{33}$$

This implies that, from the joint probabilities for a given measurement, one can obtain the correlation functions for different measurement configurations and thus the Bell function (31).

A Bell function has its boundary which is allowed by a local realistic model. It is worth mentioning that quantum mechanically correlated states do not violate the boundaries of all the possible Bell functions (31). Only the Bell functions whose boundaries are violated by quantum mechanically correlated states are of interest in the test of nonlocality [17]. In this paper, we do not try to find all the classical boundaries. Instead, we consider the Bell function whose classical upper bound is 2 with the particular vector $\vec{c} = (1, 1, 1, -1)$,

$$B = E(\vec{A}_1, \vec{B}_1) + E(\vec{A}_2, \vec{B}_2) + E(\vec{B}_2, \vec{A}_1) - E(\vec{A}_2, \vec{B}_1). \tag{34}$$

After a little algebra, one realizes that the Bell function (34) is exactly the same as the Bell function of Collins–Gisin–Linden–Massar–Popescu (CGLMP) [3], whose classical bounds are found as 2 with help of joint probabilities.

Note that the third term of the correlation function in equation (34) has the parameters for the measurement configurations exchanged. In general, the correlation function $E(\vec{p}, \vec{q})$ depends on exchanging the parameter vectors,

$$E(\vec{p}, \vec{q}) \neq E(\vec{q}, \vec{p}). \tag{35}$$

The correlation function is invariant for the parameter exchange only for dichotomic measurements in which case the Bell function (34) leads to the CHSH Bell's inequality [18].

In order to find the quantum mechanical maximum for the Bell function, equation (34), CGLMP used the QFT unitary transformation,

$$\hat{U}_{\mathrm{QFT}}(\vec{A}) = \frac{1}{\sqrt{d}} \sum_{j,k} \mathrm{e}^{\mathrm{i}\frac{2\pi}{d} j(k+\phi_A)} |j\rangle\langle k|, \tag{36}$$

which has only a single parameter ϕ_A to be adjusted for the measurement configuration. They found that for the d-dimensional *maximally* entangled state their Bell function, which is the same as equation (34), has its maximum

$$B_d = 4d \sum_{l=0}^{d-1} \left(1 - \frac{2l}{d-1}\right) \frac{1}{2d^3 \sin^2[\pi(l+1/4)/d]} \tag{37}$$

when $(\phi_{A_1}, \phi_{A_2}, \phi_{B_1}, \phi_{B_2}) = (0, 1/2, 1/4, -1/4)$. This is always larger than the local realistic upper bound 2 and increases as the number, d, of measurement outcomes increases. However, as a special subset of the unitary group U(d), it is unclear whether the QFT measurement is optimal for the test of nonlocality when $d > 3$ even though it has been known that this is the case for maximally entangled states of $d = 2$ and $d = 3$ [2].

The raised question becomes rather dramatic if a state is partially entangled. For example, when $d = 2$, the violation of Bell's inequality in equation (31) is plotted in figure 1. Note that Bell's inequality becomes the CHSH Bell's inequality when $d = 2$. The state is assumed to be in a pure state of $|\psi\rangle = \cos\varphi|00\rangle + \sin\varphi|11\rangle$. The Bell functions are optimized for the different measurement configurations: the dashed line is obtained by the QFT and the solid line by the $SU(2)$ transformations. The figure shows that the QFT is not an optimal transformation in revealing the nonlocality of the partially entangled state. It is required to consider the general $SU(d)$ transformations for the optimal nonlocality test when a state is in a partially entangled state.

3. Highly degenerate measurement for a CV state

In this section, we consider a CV state as a system to test its nonlocality. For the purpose of the nonlocality test, one needs to introduce a proper measurement which can show the
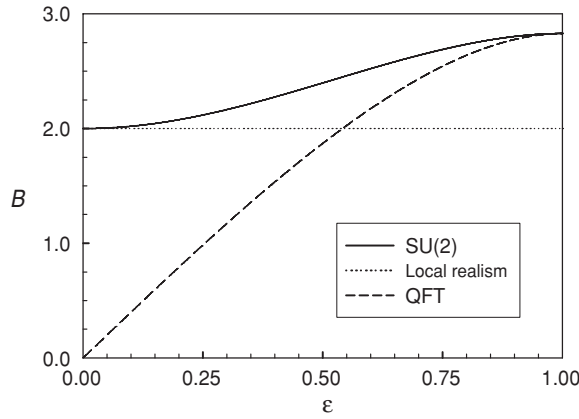
Figure 1. Bell function B with respect to the entanglement $\varepsilon = \sqrt{2}\sin\varphi$ for a two-dimensional bipartite state. Solid line represents for the case of $SU(2)$ measurement and dashed line for the measurement with QFT.

violation of any Bell's inequality. Generally, for the case of a CV state, the spectrum of its measurement is continuous and the number of nondegenerate eigenvalues is infinite. Therefore, a difficulty arises, even in principle, in measuring the infinite number of outcomes and the test of its nonlocality. Several possible methods which can overcome such a difficulty have been suggested [8, 10, 12]. These methods have adopted measurements which give a finite number of outcomes from a CV state. This measurement naturally assumes an infinite degeneracy in the measurement. Recently, it is also found that homodyne measurement after single photon subtraction from a CV state can play an essential role for a loophole-free nonlocality test [19].

In this section, we formulate the explicit form of an observable $\hat{A}$ which can give a finite d-number of outcomes from the measurement on a CV system. The observable corresponds to a mapping from a CV state to an arbitrary d-dimensional system and the mapping is the same mapping which was suggested by Brukner *et al* [12]. We show, here, that the mapping is a linear, trace preserving and complete positive (CP) map which implies that the density matrix of a CV state can be legitimately transformed into a finite-dimensional state.

3.1. d-outcome measurement for a CV system

An observable $\hat{A}(\vec{a})$ which gives d outcomes from the measurement on a CV state can be found as a direct sum of the infinite number of the d-dimensional observables;

$$\hat{A}(\vec{a}) = \begin{pmatrix} \hat{\Omega}(\vec{a}) & 0 & 0 & \cdots \\ 0 & \hat{\Omega}(\vec{a}) & 0 & \cdots \\ 0 & 0 & \hat{\Omega}(\vec{a}) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{38}$$

where $\hat{\Omega}(\vec{a})$ is the observable which is in the d-dimensional Hilbert space and has the explicit form as was given in equation (10). The measurement with the observable $\hat{A}(\vec{a})$ produces infinite degeneracy in each outcome since it counts every d modulo basis state as the same outcome. Alternatively, the observable $\hat{A}(\vec{a})$ can be written as

$$\hat{A}(\vec{a}) = \sum_{m=0}^{\infty} \sum_{jk=0}^{d-1} \Omega_{jk}(\vec{a}) |dm + j\rangle \langle dm + k|, \tag{39}$$

where $\Omega_{jk}(\vec{a})$ is the matrix element of the d-dimensional observable which is parametrized with the $d^2 - 1$ dimensional generalized Bloch vector $\vec{a}$.

With the observable $\hat{A}(\vec{a})$, one can establish the mapping between a CV state and an arbitrary finite-dimensional state for the CV state. It exploits the fact [12] that, from the physical perspective, any two systems can be considered as equivalent, if the probabilities for outcomes of all possible future experiments performed on one and on the other are the same. Mathematically, the requirement can be expressed as

$$\text{Tr}(\hat{A}(\vec{a})\hat{\rho}) = \text{Tr}(\hat{\Omega}(\vec{a})\hat{\rho}_d), \tag{40}$$

where $\hat{\rho}$ is the density matrix of any CV state while $\hat{\rho}_d$ is that of a d-dimensional state.

Moreover, it is important to clarify that the mapping is a physically possible quantum process. One can show that the observable $\hat{A}(\vec{a})$ on a CV state results in a trace preserving, linear, CP map ε as

$$\hat{\rho} \rightarrow \hat{\rho}_d = \varepsilon(\hat{\rho}), \tag{41}$$

where $\hat{\rho} \in \mathcal{B}(\mathcal{H})$ and $\hat{\rho}_d \in \mathcal{B}(\mathcal{K})$. We denote that $\mathcal{B}(\mathcal{H})$ is the set of operators defined in $\mathcal{H}$. Also note that $\mathcal{H}$ and $\mathcal{K}$ are the infinite and d-dimensional Hilbert spaces, respectively. In order to prove it, it is possible to make use of the correspondence between the complete positive maps and positive-semidefinite operators [20]. The density matrix in $\mathcal{B}(\mathcal{K})$ can be expressed by the transformation ε as follows:

$$\hat{\rho}_d = \text{Tr}_{\mathcal{H}}(\mathbb{1}_{\mathcal{K}} \otimes \hat{\rho}^T \hat{R}_\varepsilon), \tag{42}$$

where $\hat{R}_\varepsilon$ is a positive-semidefinite operator defined in $\mathcal{B}(\mathcal{K} \otimes \mathcal{H})$. The positive-semidefinite operator has the explicit form as

$$\hat{R}_\varepsilon = \sum_{n=0}^{\infty} \sum_{kl=0}^{d-1} |k\rangle\langle l| \otimes |dn+k\rangle\langle dn+l| \tag{43}$$

which satisfies the trace preserving properties of the CP map by $\text{Tr}_{\mathcal{K}}(\hat{R}_\varepsilon) = \mathbb{1}_{\mathcal{H}}$. The correspondence between the CP map and the observable $\hat{A}(\vec{a})$ is confirmed from the dual map $\varepsilon^\vee$ of the map ε [21] on the observable $\hat{\Omega}(\vec{a})$ and it is

$$\begin{aligned} \varepsilon^\vee(\hat{\Omega}(\vec{a})) &\equiv \text{Tr}_{\mathcal{K}}\left(\hat{\Omega}(\vec{a}) \otimes \mathbb{1}_{\mathcal{H}} \hat{R}_\varepsilon^{T_{\mathcal{H}}}\right) \\ &= \hat{A}(\vec{a}), \end{aligned} \tag{44}$$

where $T_{\mathcal{H}}$ denotes partial transposition on the Hilbert space $\mathcal{H}$ only. We conclude that the measurement with the observable $\hat{A}(\vec{a})$ on a CV state is equivalent to consider the state as a d-dimensional state, which is mapped from the CV state, with the measurement of $\hat{\Omega}(\vec{a})$. It can also be said that the mapping is a linear, trace preserving CP map from equation (42).

3.2. Mapping of a multi-mode state

The mapping $\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \rightarrow \mathcal{B}(\mathcal{K}_1 \otimes \mathcal{K}_2)$ for a two-mode CV density matrix onto a bipartite d-dimensional state is possible as

$$\hat{\rho}_{12} = \text{Tr}_{AB}\left(\mathbb{1}_{\mathcal{K}} \otimes \mathbb{1}_{\mathcal{K}} \otimes \hat{\rho}_{AB}^T \hat{R}_{A1}\hat{R}_{B2}\right). \tag{45}$$

The mapping for an arbitrary number of modes can also be found as an extension of equation (45).

As an example, we consider a two-mode squeezed state $|\psi\rangle$ which can be generated by a nondegenerate optical parametric amplifier [22],

$$|\psi\rangle = \sum_{n=0}^{\infty} \frac{(\tanh r)^n}{\cosh r} |n, n\rangle_{A,B}, \tag{46}$$

where $|n\rangle$ is a Fock state and r is the squeezing parameter. It is well known that when squeezing parameter r goes to infinity, the two-mode squeezed state approaches to the EPR state [7]. From equation (45), one can map the two-mode squeezed state onto the d-dimensional pure state:

$$|\psi_d\rangle = \frac{\mathrm{sech}\, r}{\sqrt{1 - \tanh^{2d} r}} \sum_{n=0}^{d-1} (\tanh r)^n |n, n\rangle_{A,B}. \tag{47}$$

The mapped state is a partially entangled pure state whose entanglement is characterized by the squeezing parameter r. The state becomes separable only when $r = 0$ and it becomes maximally entangled for the limit of $r \to \infty$.

## 4. Numerical analysis based on $SU(d)$ group

We investigate the optimal violation of Bell's inequality based on the Bell function B_d in equation (34) for the two-mode squeezed state. In order to search for optimization values of the inequalities, we employ several numerical methods such as steepest descent, conjugate gradient and dynamic relaxation. Each method has its own advantages and disadvantages depending on the situations for optimization. The conjugate gradient leads to rapid convergence for a nearly hyperbolic function (where a bounded function looks like near its minimum). The steepest descent method enables one to find persistently lower values, even though it has disadvantages of slow convergence for the nearly hyperbolic function that is squeezed in parameter space. The dynamic relaxation method is between the two methods. We consider the dynamic relaxation method in detail as the algorithms and implementations of the other methods can easily be found in literatures [23].

The dynamic relaxation method simulates a physical system under a potential and a friction, which resembles the Car–Parrinelo method for *ab initio* molecular dynamics [24]. Consider a bounded function $B(\{p_i\})$ in terms of the parameter vector p_i. For an optimization the method simulates a dynamic equation for a fictitious classical particle, by regarding p_i as its trajectory vector and $B(\{p_i\})$ as a potential. The dynamic equation of motion can be written as

$$m \frac{\mathrm{d}^2}{\mathrm{d}t^2} p_i(t) = -\gamma \frac{\mathrm{d}}{\mathrm{d}t} p_i(t) - \frac{\partial}{\partial p_i} B(\{p_j\}; t), \tag{48}$$

where m is a mass of the fictitious particle and γ is a friction ratio. Note that the equation is a kind of the Langevin equation. The particle will relax to the minimum of the potential. The solution to equation (48) approaches to the minimum of the function $B(\{p_i\})$ exponentially due to the friction γ. A minimum is claimed to be achieved when $|\partial_{p_i} B(\{p_j\})| \times |p_i| \leqslant 10^{-6}$. For the numerical implementation, a Runge–Kutta method is used to solve the dynamic equation with the following ranges of parameters: $m = 0.1$, $\gamma \in (0.5, 1.5)$ and $\delta t \in (0.01, 0.1)$. The maximum value of $B(\{p_i\})$ is obtained by replacing the 'potential' $B(\{p_i\})$ with $-B(\{p_i\})$. For the optimizations of Bell functions in equation (34), the parameter vector is given by $\vec{a} = (\vec{A}_1, \vec{A}_2, \vec{B}_1, \vec{B}_2)$ where $\vec{A}_i$ and $\vec{B}_i$ are parameter vectors for unitary transformations $\hat{U}(\vec{A}_i)$ and $\hat{U}(\vec{B}_i)$, respectively, in the group $SU(d)$.

We optimize the value of the Bell function B_d in equation (34) with $SU(d)$ transformations for the two-mode squeezed state. All the results are checked and they are reproduced by conjugate gradient, steepest descent and dynamic relaxation methods. Figure 2 presents the optimized value of the Bell functions B_d with respect to the strength of squeezing $\tanh(r)$ for different numbers of measurement outcomes. The Bell function is upper bounded by 2, $B_d \leqslant 2$, under the local realistic theory.
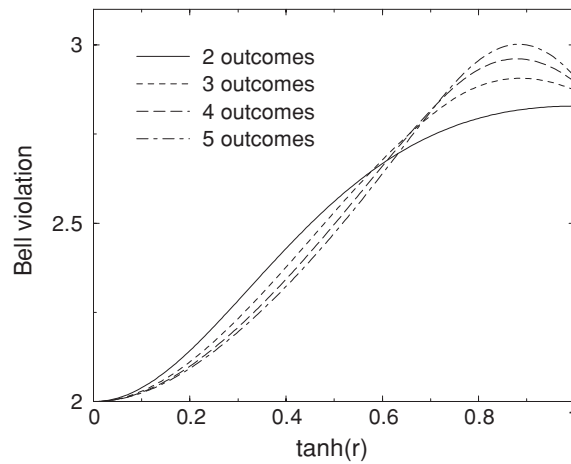
Figure 2. Violation of Bell's inequality based on the Bell function B_d for a two-mode squeezed state. Finite number of measurement outcomes are considered.

Table 1. The largest optimum values, $B_d(r_m)$, of the Bell function for a two-mode squeezed state. r_m is the squeezing parameter which maximized the value of B_d for the given number, d, of measurement outcome. $B_d(\infty)$ is the value of B_d for an infinite squeezing.

d	r_m	$B_d(r_m)$	$B_d(\infty)$
2	∞	2.828 43	2.828 43
3	1.407	2.906 38	2.872 93
4	1.373	2.960 95	2.896 24
5	1.393	3.001 87	2.910 55

In figure 2, we note that a two-mode squeezed state always violates the inequality $B_d \leqslant 2$ for all $r > 0$ regardless of the number of measurement outcomes. Brukner *et al* calculated the values of the Bell function B_d for $d = 3$ based on the QFT for a two-mode squeezed state. They do not always achieve $B_d > 2$ for the squeezing parameter $r > 0$ even though a two-mode squeezed state is inseparable. Thus in order to properly achieve the optimum value of the Bell function, we have to consider all the possible transformations in $SU(d)$.

On the other hand, in the limit of $r \to \infty$, a two-mode squeezed state becomes a regularized EPR state which is mapped onto a maximally entangled state in finite-dimensional Hilbert space. However, for the maximally entangled state, the QFT suffices to obtain the optimum value of B_d as we have already discussed.

For a given number of measurement outcomes, the amount of violation increases and decreases with respect to the squeezing parameter r. Let r_d denote the value of squeezing parameter that gives the largest violation for a given measurement with d number of outcomes. As shown in table 1, except for the dichotomic measurement, the infinitely squeezed state violates the inequality less than some partially entangled states. As one increases the number of outcomes, the largest optimum values of B_d monotonically increase. It is also found in figure 2 that for the high squeezing regime the higher number of outcomes gives stronger violation while the result is reversed for the small squeezing regime.

5. Final remarks

We studied the most general d-outcome measurement for Bell's inequalities of a bipartite system. In order to construct the inequalities, we introduced a classically correlated observable which is constructed in terms of local measurements and classical communications. For the configuration of the local measurements, we considered general transformations in $SU(d)$. It was found that the number of parameters for the nontrivial operation is reduced to $(d^2 - d)$. After inspection of symmetries, we derived the Bell function that is composed of the correlation functions. This Bell function is equivalent to that found by Collins–Gisin–Linden–Massar–Popescu [3]. The present numerical analysis shows that, when the system is in a maximally entangled state, the QFT is an optimal transformation for each local measurement. However, we show that this does not hold when the system is in a partially entangled state.

In order to utilize the CV state for the nonlocality test, we investigated the mapping between a CV state and an arbitrary dimensional system which was devised by Brukner *et al* [12]. We found the mapping is a linear, trace preserving and complete positive map and it corresponds to a highly degenerate d-outcome measurement on a CV state. By applying the highly degenerate measurements, we investigated the optimal violation of Bell's inequality for the two-mode squeezed state. Regardless of the degree of squeezing and the number of outcomes, the two-mode squeezed state always violates Bell's inequalities. This opens a possibility of extending Gisin's theorem [13], which states that a pure entangled bipartite system always shows nonlocality not only for the case of dichotomic measurement but also for the case of a measurement with an arbitrary number of measurement outcomes.

Acknowledgments

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